Generating Functions and Stability Study of Multivariate Self-Excited Epidemic Processes

A. Saichev[1,2] and D. Sornette[1]

[1]*Department of Management, Technology and Economics, ETH Zurich, Kreuzplatz 5, CH-8032 Zurich, Switzerland*

[2]*Mathematical Department, Nizhny Novgorod State University, Gagarin prosp. 23, Nizhny Novgorod, 603950, Russia**

(Dated: January 31, 2011)

Abstract

We present a stability study of the class of multivariate self-excited Hawkes point processes, that can model natural and social systems, including earthquakes, epileptic seizures and the dynamics of neuron assemblies, bursts of exchanges in social communities, interactions between Internet bloggers, bank network fragility and cascading of failures, national sovereign default contagion, and so on. We present the general theory of multivariate generating functions to derive the number of events over all generations of various types that are triggered by a mother event of a given type. We obtain the stability domains of various systems, as a function of the topological structure of the mutual excitations across different event types. We find that mutual triggering tends to provide a significant extension of the stability (or subcritical) domain compared with the case where event types are decoupled, that is, when an event of a given type can only trigger events of the same type.

*Electronic address: saichev@hotmail.com,dsornette@ethz.ch

I. INTRODUCTION

Many natural and social systems are punctuated by short-lived events that play a particularly important role in their organization. Such events can be conveniently modeled mathematically by so-called point processes [1, 2]. They are also called shot noise in physics [3–5] or jump processes in finance and in economics [6]. These models are characterized by their (conditional) rate $\lambda(t|H_t)$ (also called "conditional intensity") defined as the limit for small time intervals Δ of the probability that an event occurs between t and $t + \Delta$, given the whole past history H_t. In mathematical notations, this reads

$$\lambda(t|H_t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Pr(\text{event occurs in } [t, t + \Delta]|H_t) \ , \tag{1}$$

where $\Pr(X|H_t)$ represents the probability that event X occurs, conditional on the past history H_t. The symbol H_t represents the entire history up to time t, which includes all previous events. This definition is straightforward to generalize for space-dependent intensities $\lambda(t, \vec{r}|H_t)$ and to include marks such as amplitudes or magnitudes (see below). The standard Poisson memoryless process is the special case such that $\lambda(t|H_t)$ is constant, i.e., independent of the past history. Clustered point processes generalize the Poisson process by assuming that the series of events are generated from a cluster center process, which is often a renewal process, and a cluster member process.

The class of point processes that we study here was introduced by Hawkes in 1971 [7–10]. It is much richer and relevant to most natural and social systems, because it describes "self-excited" processes. This term means that the past events have the ability to trigger future events, i.e., $\lambda(t|H_t)$ is a function of past events, being therefore non-markovian. Many works have been performed to characterize the statistical and dynamical properties of this class of models, with applications ranging from geophysical [11–16], medical [17] to financial systems, with applications to Value-at-risk modeling [18], high-frequency price processes [19], portfolio credit risks [20], cascades of corporate defaults [21], financial contagion [22], and yield curve dynamics [23].

While surprisingly rich and powerful in explaining empirical observations in a variety of systems, most previous studies have used mono-variate self-excited point processes, i.e., they have assumed the existence of only a single type of events, all the events presenting some ability to trigger events of the same type. However, in reality, in many systems, events come

in different types with possibly different properties, while keeping a degree of mutual inter-excitations. Among others, this applies to geo-tectonic deformations and earthquakes, to neuronal excitations in the brain, to financial volatility bursts in different assets, to defaults on debts in some firms or some industrial sectors, to sovereign risks in some countries within a currency block, to the heterogeneity of activity of bloggers on the Internet, and so on.

These observations suggest that multivariate self-excited point processes, which extend the class of mono-variate self-excited point processes, provide a very important class of models to describe the self-excitation (or intra-triggering) as well as the mutual influences or triggering between different types of events that occur in many natural and social systems. These considerations have motivated us to present recently the first exact analysis of some of the temporal properties of multivariate self-excited Hawkes conditional Poisson processes [24], as they constitute powerful representations of a large variety of systems with bursty events, for which past activity triggers future activity. The term "multivariate" refers here to the property that events come in different types, with possibly different intra and inter-triggering abilities. Ref. [24] was a first step towards a systematic study of the multivariate self-excited point processes, first mentioned by Hawkes himself in his first paper [7], whose full relevance has only been recently appreciated [22, 25].

The present paper is a complementary study to our previous paper [24], which was focused on temporal properties, by studying the general stability conditions of this class of models. Section 2 recalls the definition and notation of Hawkes processes, starting from the monovariate version and extending to the general multivariate formulation. Section 3 presents the formalism of multivariate generating functions to derive the number of events over all generations of various types that are triggered by a mother event of a given type. Section 4 gives the stability conditions using the mean numbers of events of all generations. Subsection 4A provides the general relations. Subsection 4B studies the case of symmetric mutual excitation abilities between events of different types. Subsection 4C restricts to the case of just two different types of events, that allows an in-depth analysis of the new features resulting from the inter-type excitations. Subsection 4D presents the results obtained for a one-dimensional chain of directed triggering in the space of event types. Subsection 4E generalizes subsection 4D by studying a one-dimensional chain in the space of event types with nearest-neighbor triggering. Subsection 4F presents a quantitative measure of the size of the subcritical domain that allows us to study the influence of the inter-type coupling

strength. Section 5 concludes.

II. DEFINITIONS AND NOTATIONS FOR THE MULTIVARIATE HAWKES PROCESSES

A. Monovariate Hawkes processes

Self-excited conditional Poisson processes generalize the cluster models by allowing each event, including cluster members, i.e., aftershocks, to trigger their own events according to some memory kernel $h(t - t_i)$.

$$\lambda(t|H_t, \Theta) = \lambda_c(t) + \sum_{i|t_i<t} h(t - t_i,) \ , \tag{2}$$

where the history $H_t = \{t_i\}_{1 \leq i \leq i_t,\ t_{i_t} \leq t < t_{i_t+1}}$ includes all events that occurred before the present time t and the sum in expression (2) runs over all past triggered events. The set of parameters is denoted by the symbol Θ. The term $\lambda_c(t)$ means that there are some external background sources occurring according to a Poisson process with intensity $\lambda_c(t)$, which may be a function of time, but all other events can be both triggered by previous events and can themselves trigger their offsprings. This gives rise to the existence of many generations of events.

Introducing "marks" or characteristics for each event leads to a first multidimensional extension of the self-excited process (2). The generalization consists in associating with each event some marks (possible multiple traits), drawn from some distribution $p(m)$, usually chosen invariant as a function of time:

$$\lambda(t, M|H_t, \Theta) = p(M) \left(\lambda_c(t) + \sum_{i|t_i<t} h(t - t_i, M_i) \right) \ , \tag{3}$$

where the mark M_i of a given previous event now controls the shape and properties of the triggering kernel describing the future offsprings of that event i. The history now consists in the set of occurrence times of each triggered event and their marks: $H_t = \{t_i, M_i\}_{1 \leq i \leq N}$. The first factor $p(M)$ in the r.h.s. of expression (3) writes that the marks of triggered events are drawn from the distribution $p(M)$, independently of their generation and waiting times. This is a simplifying specification, which can be relaxed. Inclusion of spatial kernel to describe how distance impacts triggering efficiency is straightforward.

From a theoretical point of view, the Hawkes models with marks has been studied in essentially two directions: (i) statistical estimations of its parameters with corresponding residual analysis as goodness of fits [39–48]; (ii) statistical properties of its space-time dynamics [15, 16, 29–38].

The advantage of the self-excited conditional Hawkes process includes a very parsimonious description of the complex spatio-temporal organization of systems characterized by self-excitation of "bursty" events, without the need to invoke ingredients other than the generally well-documented stylized facts on the distribution of event sizes, the temporal "Omori law" for the waiting time before excitation of a new event and the productivity law controlling the number of triggered events per initiator.

Self-excited models of point processes with additive structure of their intensity on past events [10] make them part of the general family of branching processes [49]. The crucial parameter is then the branching ratio n, defined as the mean number of events of first generation triggered per event. Depending on applications, the branching ratio n can vary with time, from location to location and from type to type (as we shall see below for the multivariate generalization). The branching ratio provides a diagnostic of the susceptibility of the system to trigger activity in the presence of some exogenous nucleating events.

We refer in particular to Ref. [17] for a short review of the main results concerning the statistical properties of the space-time dynamics of self-excited marked Hawkes conditional Poisson processes.

B. Multivariate Hawkes processes

The Multivariate Hawkes Process generalizes expressions (3) into the following general form for the conditional Poisson intensity for an event of type j among a set of m possible types (see the document [26] for an extensive review):

$$\lambda_j(t|H_t) = \lambda_j^0(t) + \sum_{k=1}^{m} \Lambda_{kj} \int_{(-\infty,t)\times\mathcal{R}} h_j(t-s)\ g_k(x)\ N_k(ds\times dx)\ , \tag{4}$$

where H_t denotes the whole past history, λ_j^0 is the rate of spontaneous (exogenous) events of type j, sources of immigrants of type j, Λ_{kj} is the (k,j)'s element of the matrix of coupling between the different types which quantifies the ability of a type k-event to trigger a type j-event. Specifically, the value of an element Λ_{jk} is just the average number of first-generation

events of type j triggered by an event of type k. The memory kernel $h_j(t-s)$ gives the probability that an event of type k that occurred at time $s < t$ will trigger an event of type j at time t. The function $h_j(t-s)$ is nothing but the distribution of waiting times (here between the impulse of event k which impacted the system at time s, the system taking a certain time $t-s$ to react with an event of type j, this time being a random variable distributed according to the function $h_j(t-s)$. The fertility (or productivity) law $g_k(x)$ of events of type k with mark x quantifies the total average number of first-generation events of any type triggered by an event of type k. We have used the standard notation $\int_{(-\infty,t)\times\mathcal{R}} f(t,x)N(ds\times dx) := \sum_{k|t_k<t} f(t_i,x_i)$.

The matrix Λ_{kj} embodies both the topology of the network of interactions between different types, and the coupling strength between elements. In particular, Λ_{kj} includes the information contained on the adjacency matrix of the underlying network. Analogous to the condition $n < 1$ (subcritical regime) for the stability and stationarity of the monovariate Hawkes process, the condition for the existence and stationarity of the process defined by (4) is that the spectral radius of the matrix Λ_{kj} be less than 1. Recall that the spectral radius of a matrix is nothing but its largest eigenvalue.

III. MULTIVARIATE GENERATING FUNCTION (GF)

A. Definition for events of first-generation events triggered by a given mother of type k

Among the m types of events, consider the k-th type and its first generation offsprings. Let us denote $R_1^{k,1}, R_1^{k,2}, \dots, R_1^{k,m}$, the number of "daughter" events of first generation of type $1, 2, \dots, m$ generated by this "mother" event of type k. With these notations, the generating function (GF) of all events of first generation that are triggered by a mother event of type k reads

$$A_1^k(y_1, y_2, \dots, y_m) := \mathrm{E}\left[\prod_{s=1}^{m} y_s^{R_1^{k,s}}\right] , \tag{5}$$

where $\mathrm{E}[.]$ represents the statistical average operator. One may rewrite this function in probabilistic form

$$A_1^k(y_1, y_2, \dots, y_m) := \sum_{r_1=0}^{\infty} \cdots \sum_{r_m=0}^{\infty} P_k(r_1, \dots, r_m) \prod_{s=1}^{m} y_s^{r_s} , \tag{6}$$

where $P_k(r_1, \ldots, r_m)$ is the probability that the mother event of type k generates $R^{k,1} = r_1$ first-generation events of type 1, $R^{k,2} = r_2$ first-generation events of type 2, and so on. These probabilities satisfy to normalizing condition

$$\sum_{r_1=0}^{\infty} \cdots \sum_{r_m=0}^{\infty} P_k(r_1, \ldots, r_m) = 1 \ . \tag{7}$$

The first-order moments or mean values of the numbers of first-generation events of different types triggered by a mother of type k are given by

$$n_{k,s} = \frac{\partial}{\partial y_s} A_1^k(y_1, y_2, \ldots, y_m)\big|_{y_1=\cdots=y_m=1} \tag{8}$$

B. Generating function (GF) for all-generation events triggered by a given mother of type k

The GF $A^k(y_1, y_2, \ldots, y_m)$ for all-generation events triggered by a given mother of type k is by definition equal to

$$A^k(y_1, y_2, \ldots, y_m) := \mathrm{E}\left[\prod_{s=1}^{m} y_s^{R^{k,s}}\right] \ , \tag{9}$$

where $R^{k,1}, R^{k,2}, \ldots, R^{k,m}$ are the numbers of events of all generations and all kinds that are triggered by the mother event of type k.

In order to relate $A^k(y_1, y_2, \ldots, y_m)$ to $A_1^k(y_1, y_2, \ldots, y_m)$, we assume that the first-generation daughters can also trigger their own daughters (which are the grand-daughters of the initial event) according to the following rules.

- The numbers of second-generation events that are triggered by each first-generation event are statistically independent of the numbers of first-generation events. They are also statistically independent of the numbers of second-generation events that are triggered by any other first-generation events.

- Each first-generation event triggers second-generation events according to the same laws controlling the triggering of first-generation events by the initial mother event of the same type. In other words, the same laws apply to the generation of new events from generation to generation, independently of the generation depth.

These rules allow us to derive the GF of the numbers of first-generation and of second-generation events by performing the following replacement for each variables y_q in the expression of the GF $A_1^k(y_1, y_2, \ldots, y_m)$:

$$y_q \qquad \rightarrow \qquad y_q \cdot A_1^q(y_1, y_2, \ldots, y_m) \ . \tag{10}$$

The GF of the numbers of first-generation and of second-generation events that are triggered by a mother event of type k is given by the following expression in terms of the GF $A_1^k(y_1, y_2, \ldots, y_m)$ of the numbers of first-generation events that are triggered by a mother event of type k:

$$A_2^k(y_1, y_2, \ldots, y_m) = A_1^k(y_1 \cdot A_1^1(y_1, y_2, \ldots, y_m), \ldots, y_m \cdot A_1^m(y_1, y_2, \ldots, y_m)) \ . \tag{11}$$

This equation is valid for all possible values of $k = 1, \ldots, m$.

By recurrence, one obtain the GF A_{j+1}^k of the numbers of events of all generations up to $j+1$ that are triggered by an initial mother event of type k as a function of the GF $\{A_j^k\}$ of the numbers of events of all generations up to j triggered by an initial mother event of type k:

$$A_{j+1}^k(y_1, y_2, \ldots, y_m) = A_1^k(y_1 \cdot A_j^1(y_1, y_2, \ldots, y_m), \ldots, y_m \cdot A_j^m(y_1, y_2, \ldots, y_m)) \ . \tag{12}$$

This equation is valid for all possible values of $k = 1, \ldots, m$ and for all possible generation levels $j = 2$ to $+\infty$.

We assume that the above set of recurrence equations (12) for $k = 1, \ldots, m$ converges to some set of GF's $\{A^k(y_1, y_2, \ldots, y_m); k = 1, \ldots, m\}$. Then, the corresponding GF's $\{A^k(y_1, y_2, \ldots, y_m); k = 1, \ldots, m\}$ are solutions of the transcendent equations

$$A^k(y_1, y_2, \ldots, y_m) = A_1^k(y_1 \cdot A^1(y_1, y_2, \ldots, y_m), \ldots, y_m \cdot A^m(y_1, y_2, \ldots, y_m)) \ , \ k = 1, \ldots, m \ . \tag{13}$$

The equation constitutes the basis for our subsequent analysis.

IV. STABILITY CONDITIONS USING MEAN NUMBERS OF EVENTS OF ALL GENERATIONS

A. General relations

The statistical average of the total numbers of events of type s over all generations that are triggered by a mother of type k is given by

$$\bar{R}^{k,s} = \frac{\partial}{\partial y_s} A^k(y_1, y_2, \dots, y_m)\big|_{y_1=y_2=\dots=y_m=1} . \tag{14}$$

Using (13), it is straightforward to show that $\bar{R}^{k,s}$ is solution of

$$\bar{R}^{k,s} = n_{k,s} + \sum_{\ell=1}^{m} n_{k,\ell} \cdot \bar{R}^{\ell,s} , \tag{15}$$

where $n_{k,s}$ is the mean number of first-generation events of type s triggered by the mother event of type k.

Since expression (15) holds for all $k = 1, \dots, m$ and $s = 1, \dots, m$, it can written in matrix form

$$\hat{R} = \hat{N} + \hat{N}\hat{R} , \tag{16}$$

where $\hat{N} = [n_{k,s}]$ is the matrix of the mean numbers of first-generation events and $\hat{R} = [\bar{R}^{k,s}]$ is the matrix of the mean numbers of events over all generations. The sum over row indices of the elements of the matrix $\hat{N}$

$$n_k = \sum_{s=1}^{m} n_{k,s} , \tag{17}$$

is the mean number of first-generation events of all kinds that are triggered by a mother event of type k.

The solution of the matrix equation (16) is

$$\hat{R} = \frac{\hat{N}}{\hat{I} - \hat{N}} . \tag{18}$$

The rest of the paper is concerned with the analysis of particular examples of this general solution, worked out for different systems and excitation conditions embodied in different forms of the matrix $\hat{N}$ of the mean numbers of first-generation events.

B. Symmetric mutual excitations

Let us consider the case where

$$n_{k,k} = a \; ; \qquad\qquad n_{k,s} = b, \quad k \neq s \; , \tag{19}$$

resulting in the form

$$\hat{N} = \begin{bmatrix} a \;\; b \;\; b \;\; \ldots\ldots\ldots\ldots\ldots \; b \\ b \;\; a \;\; b \;\; \ldots\ldots\ldots\ldots\ldots \; b \\ b \;\; b \;\; a \;\; \ldots\ldots\ldots\ldots\ldots \; b \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \underbrace{b \ldots\ldots\ldots\ldots\ldots\ldots \; b \;\; b \;\; a}_{m} \end{bmatrix} \tag{20}$$

for the matrix $\hat{N}$ of the mean numbers of first-generation events. This form (19) means that events of a given type have identical triggering efficiencies quantified by a to generate first-generation events of the same type. They also have identical efficiencies quantified by b to trigger first-generation events of a different type. In other words, the mean number of first-generation events of type k triggered by a mother event of the same type k is independent of k. And the mean number of first-generation events of any type $s \neq k$ triggered by any another event of a different type k is independent of k and s.

As a consequence, the mean number of first-generation events of all kinds that are triggered by a mother event of some type k, as given by (17), is independent of k and given by

$$n_k = n = a + (m-1)b \; , \qquad \text{for all } k \; . \tag{21}$$

It is convenient to introduce the factor

$$q = \frac{b}{a} \tag{22}$$

comparing the inter-types with the intra-type triggering efficiencies. Using definition (22) and equality (21), we obtain

$$a = \frac{n}{1+(m-1)q} \; , \qquad b = \frac{nq}{1+(m-1)q} \; . \tag{23}$$

Two limiting cases are worth mentioning: $q = 0$ (independent types) and $q = 1$ (fully equivalent types):

$$a|_{q=0} = n, \quad b|_{q=0} = 0, \qquad a|_{q=1} = b|_{q=1} = \frac{n}{m} \; . \tag{24}$$

The solution (18) implies that the matrix $\hat{R}$ possesses the same structure as the matrix $\hat{N}$, with identical diagonal elements $\bar{R}^{k,k}$ and identical off-diagonal elements $\bar{R}^{k,s}$ (for $k \neq s$), given respectively by

$$\begin{aligned} \bar{R}^{k,k} &= \frac{n}{1-n} \cdot \frac{1+n(q-1)}{1+n(q-1)+q(m-1)} \ , \quad k = 1, ..., m \ , \\ \bar{R}^{k,s} &= \frac{n}{1-n} \cdot \frac{q}{1+n(q-1)+q(m-1)}, \qquad k \neq s. \end{aligned} \tag{25}$$

Therefore, the mean $\bar{R}^k$ of the total number of events of all kinds, that are triggered by some given mother event of a given type k, is given by

$$\bar{R}^k = \sum_{s=1}^{m} \bar{R}^{k,s} = \frac{n}{1-n} := \bar{R} \ , \quad \forall q \ . \tag{26}$$

This expression has a simple interpretation, resulting from equation (21) and the process of triggering. Indeed, by definition of n in (21), there are on average n first-generation events of all kinds that are triggered by a mother event of some type k. Each of these first-generation event triggers on average n second-generation events of all kinds, leading to a total contribution n^2 for the number of second-generation events. Counting all the generation cascades, we obtain $n + n^2 + n^3 + ...$, which is nothing but the result (26).

As for the mono-variate Hawkes process, the dynamics is stable (sub-critical) for $n < 1$ and unstable (super-critical or exponentially explosive) for $n > 1$. As usual, the critical point occurs when there is exactly $n = 1$ first-generation events of all kinds that are triggered by a mother event of any type. There is not qualitative difference between this multi-variate Hawkes process with the structure (20) of mutual excitations and a mono-variate Hawkes process, once the branching ratio n defined as the average number of first-generation daughters from a given mother is generalized into its natural extension (21).

C. Two-dimensional mutually and self-excited Hawkes process

With only two types of events, a detailed analysis can be performed, with the discovery of new qualitative regimes.

1. Stability analysis

With two types of events, the 2×2 matrix $\hat{N}$ of the mean numbers of of first-generation events can be kept fully general and is noted as

$$\hat{N} = \begin{bmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{bmatrix} \tag{27}$$

The solution (18) is a 2×2 matrix $\hat{R}$ with elements $\bar{R}^{k,s}$ given by

$$\begin{aligned} \bar{R}^{1,1} &= \frac{n_{1,1} + n_{1,2}n_{2,1} - n_{1,1}n_{2,2}}{\mathcal{D}}, \qquad \bar{R}^{1,2} = \frac{n_{1,2}}{\mathcal{D}}, \\ \bar{R}^{2,1} &= \frac{n_{2,1}}{\mathcal{D}}, \qquad \bar{R}^{2,2} = \frac{n_{2,2} + n_{1,2}n_{2,1} - n_{1,1}n_{2,2}}{\mathcal{D}}, \end{aligned} \tag{28}$$

where

$$\mathcal{D} = 1 + n_{1,1}n_{2,2} - n_{1,2}n_{2,1} - n_{1,1} - n_{2,2}. \tag{29}$$

In order to determine the stability of two-dimensional mutually and self-excited Hawkes process, we study the mean numbers n_1 and n_2 of first-generation events triggered by a mother event of the first and second kind, respectively. They are given by the sums of the two row elements of the matrix $\bar{N}$:

$$n_1 = n_{1,1} + n_{1,2}, \qquad n_2 = n_{2,1} + n_{2,2}. \tag{30}$$

It is convenient to use a representation of the elements $n_{k,s}$ of the matrix $\hat{N}$ similar to (23):

$$n_{1,1} = \frac{n_1}{1+q_1}\ , \quad n_{1,2} = \frac{n_1 q_1}{1+q_1}\ , \quad n_{2,1} = \frac{n_2 q_2}{1+q_2}\ , \quad n_{2,2} = \frac{n_2}{1+q_2}\ . \tag{31}$$

As in (22), q_1 and q_2 quantify the relative strengths of inter-type compared with the intra-type triggering efficiencies: $q_1 = n_{1,2}/n_{1,1}$ and $q_2 = n_{2,1}/n_{2,2}$. The limit $q_1 = q_2 = 0$ reduces to two independent self-excited Hawkes processes.

The solutions $\bar{R}^{k,s}$ given by (28) are finite as long as the spectral radius $\lambda(n_1, n_2)$ of the matrix $\bar{N}$ remains smaller than 1. This defines the sub-critical regime. The system becomes critical (respectively super-critical) when the spectral radius $\lambda(n_1, n_2)$ is equal to 1 (respectively larger than 1). One can show that the set $(n_1; n_2)$ such that the denominator $\mathcal{D}$ given by (29) is identically zero is critical, i.e., corresponds to a unit spectral radius, if $\mathcal{D}$ remains positive in the domain bounded by the semi-axes $[n_1 \in (0; +\infty), n_2 \in (0; +\infty)]$ and the curve $\mathcal{D}(n_1; n_2) = 0$.

In order to study the three regimes (sub-critical, crtical and super-critical), it is convenient to express $\mathcal{D}(n_1, n_2)$ as a function of n_1 and n_2, using (31):

$$\mathcal{D}(n_1, n_2) = 1 + \frac{n_1 n_2 (1 - q_1 q_2)}{(1 + q_1)(1 + q_2)} - \frac{n_1}{1 + q_1} - \frac{n_2}{1 + q_2} \; . \tag{32}$$

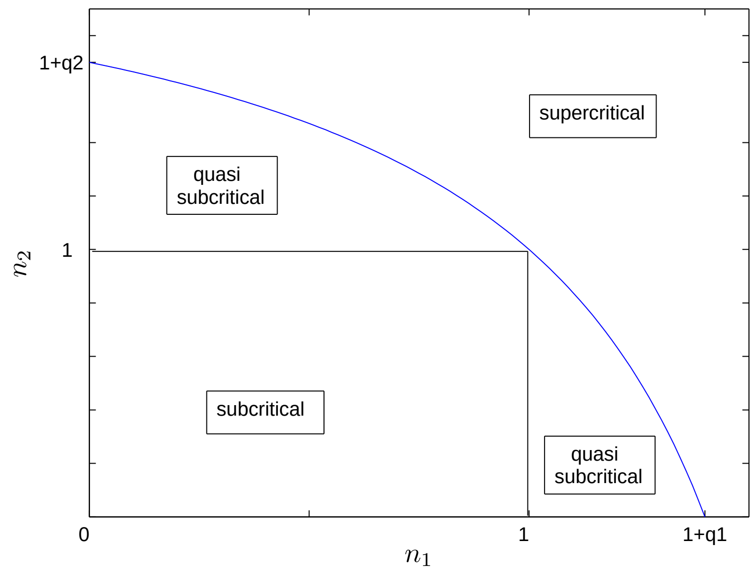

Fig. 1: Critical line $\mathcal{D}(n_1, n_2) = 0$ and three domains in the plane (n_1, n_2): 1) subcritical, where both n_1 and n_2 are smaller than 1; 2) quasi subcritical, where one of the mean numbers (n_1, n_2) of first-generation events is larger than one but the mean numbers of events of all generations are finite; 3) supercritical region, where all mean numbers $\bar{R}^{k,s}$ are infinite.

The critical line $\mathcal{D}(n_1, n_2) = 0$ is shown in figure 1 in the plane (n_1, n_2), together with three domains.

1. For $n_1 < 1$ and $n_2 < 1$, $\mathcal{D}(n_1, n_2) > 0$ and the system is subcritical. The conditions $n_1 < 1$ and $n_2 < 1$ mean that the cascade of events over all generations do not blow up for each of the two types of event triggering.

2. The domain indicated in figure 1 as "quasi subcritical" is such that one of the mean numbers (n_1, n_2) of first-generation events is larger than one but the mean numbers of events over all generations remain finite since $\mathcal{D}(n_1, n_2) > 0$. Intuitively, the super-critical regime of one of the event types is damped out by the triggering of the second type of events which is subcritical. The two extreme boundaries $(n_1 = 1 + q_1; n_2 = 0)$

and $(n_1 = 0; n_2 = 1 + q_2)$ exemplify this point as they correspond respectively to $n_{1,1} = 1; n_2 = 0$ and $n_1 = 0; n_{2,2} = 1$.

3. In the domain indicated at "supercritical" in figure 1, the mean number of events of both types summed over all generations goes to infinity. This occurs of course if both n_1 and n_2 are larger than 1 but also when one of them is smaller than 1 if the other one is sufficiently large. In this later case, the damping offered by the second type is not sufficient to stabilize the triggering process. This is the runaway explosive regime.

4. The downward sloping line defines the critical domain $\mathcal{D}(n_1, n_2) = 0$ separating the quasi subcritical and the supercritical regimes.

As the critical line $\mathcal{D}(n_1, n_2) = 0$ is approached from within the subcritical regime, the total mean number $\bar{R}^1 = \bar{R}^{1,1} + \bar{R}^{1,1}$ of events of all types and over all generations,

$$\bar{R}^1 = \bar{R}^{1,1} + \bar{R}^{1,1} = \frac{n_1(1 + q_1 + q_2 + q_1 q_2 - n_2(1 - q_1 q_2))}{(1 + q_2)(1 + q_1 - n_1) - (1 + q_1 - n_1(1 - q_1 q_2))n_2} \ , \tag{33}$$

grows to finally diverge on the line, as shown in figure 2 for a particular example.

In this example, $q_1 = 0.2, q_2 = 0.4, n_1 = 0.8$. The value $q_1/(1+q_1) = 16.7\%$ is the fraction of first-generation events generated by a mother of the first type which are of the second type. The value $q_2/(1 + q_2) = 28.6\%$ is the fraction of first-generation events generated by a mother of the second type which are of the first type.

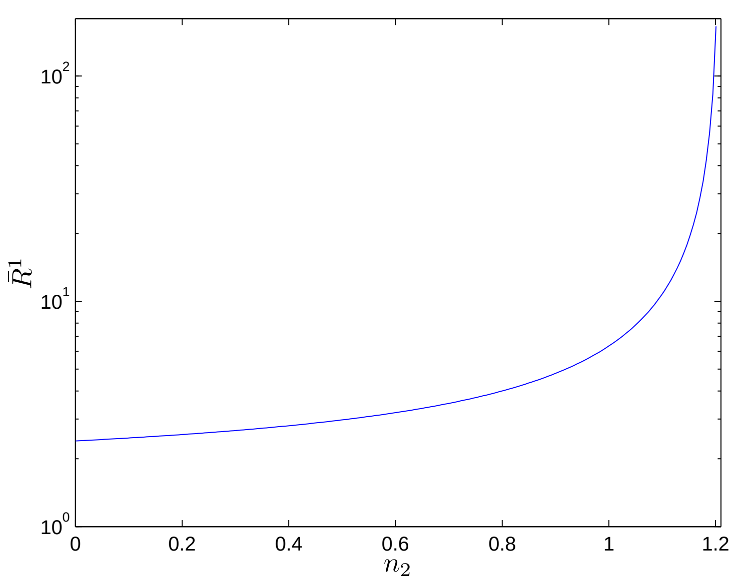

Fig. 2: Linear-log plot of the mean number $\bar{R}^1$ of events of all types and over all generations given by expression (33) as a function of n_2 for $q_1 = 0.2$, $q_2 = 0.4$ and $n_1 = 0.8$. Note that $\bar{R}^1$ remains finite even when n_2 becomes larger than 1 up to a critical value $n_2^c = 1.206896...$ for which $\mathcal{D}(n_1, n_2) = 0$ at which it diverges. For $n_2 = 1.2$ for instance, $\bar{R}^1 = 144$.

2. Strong asymmetry in mutual triggering

It is instructive to consider the limiting case where one type of events triggers many more events of the other type than the reverse. Mathematically, this corresponds to

$$q_1 \gg q_2 \ , \tag{34}$$

which means that the fraction of first-generation events generated by a mother of the first type which are of the second type is much larger than the fraction of first-generation events generated by a mother of the second type which are of the first type. In this case, the critical line $\mathcal{D}(n_1, n_2) = 0$ becomes almost rectangular, as illustrated in figure 3.

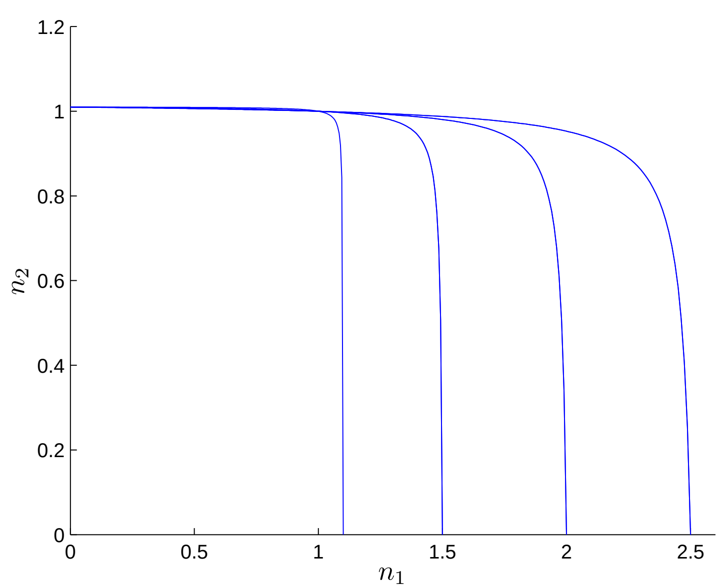

Fig. 3: Plots of the critical line $\mathcal{D}(n_1, n_2) = 0$ for the four following values of $(q_1; q_2)$: (0.1;0.01), (0.5;0.01), (1;0.01), (1.5;0.01) from left to right.

Let us consider in more details the limiting case $q_1 = q$ while $q_2 = 0$, for which the relations (31) transform into

$$n_{1,1} = \frac{n_1}{1+q} \ , \quad n_{1,2} = \frac{n_1 q}{1+q} \ , \quad n_{2,1} = 0 \ , \quad n_{2,2} = n_2 \ . \tag{35}$$

As a result, the relations (28) and (29) become

$$\begin{aligned} \bar{R}^{1,1} = \frac{n_1}{1+q-n_1}, \qquad \bar{R}^{1,2} = \frac{n_1 q}{(1+q-n_1)(1-n_2)}, \\ \bar{R}^{2,2} = \frac{n_2}{1-n_2}, \qquad \bar{R}^{2,1} = 0. \end{aligned} \tag{36}$$

The finiteness (subcritical behavior) of the number $\bar{R}^{1,1}$ is controlled solely by n_1, which must be smaller than $1+q$. As n_1 tends to $1+q$, $\bar{R}^{1,1}$ goes to infinity, expressing the transition to the supercritical regime. In contrast, there are two mechanisms leading to the divergence of $\bar{R}^{1,2}$.

1. As n_1 tends to $1+q$, $\bar{R}^{1,2}$ goes to infinity, as the number of events of the first type itself diverges, each of these events triggering a significant fraction of events of the second type, at each generation. In other words, the divergence of $\bar{R}^{1,2}$ is controlled by or slaved to that of $\bar{R}^{1,1}$, which reflects the triggering efficiency of events of type one.

2. The number $\bar{R}^{1,2}$ of events of the second type generated over all generations by a mother event of type one diverges when $n_2 \to 1$, even if $n_1 < 1+q$. A mother event of type one triggers events of type two at each generation, each of these events only triggering events of their own kind. Thus, the number of events of the second type diverges when the self-triggering parameter (or "branching ratio") n_2 reaches its critical value $n_2^c = 1$.

This implies in particular that, when q or n_2 are sufficiently large, the following inequality holds: $\bar{R}^{1,2} > \bar{R}^{1,1}$. The general condition for this to be true is $n_2 + q > 1$. Intuitively, for a fixed self-triggering ability n_2, there must be sufficiently many events of type two generated by events of type one: $q > 1 - n_2$. Alternatively, for a fixed fraction $q/(1+q)$ of first-generation events of type two generated by events of type one, the branching ratio of type two events must be sufficient large: $n_2 > 1 - q$.

Note that, if both n_{12} and n_{21} are positive, corresponding to a non-degenerate case, then events of any type do trigger events of the other type. As a consequence, either $\bar{R}^1$ and $\bar{R}^2$

are both finite or both infinite. This is not the case for independent or semi-independent systems. A system is independent if $n_{12} = n_{21} = 0$, i.e., events of different types live their separate "lives" without any inter-mutual triggering. In this case, for instance, if $n_1 < 1$, while $n_2 > 1$, then the events of the first type form a subcritical set, while the events of the second type form a supercritical system. Two systems are semi-independent if only one of the two cross-terms n_{12} and n_{21} is equal to zero. Suppose for instance that $n_{21} = 0$. Then, if $\bar{R}^{1,1}$ is finite, then $\bar{R}^{1,2}$ might be finite or infinite, just because events of the second type cannot trigger events of the second type and an infinite value of $\bar{R}^{1,2}$ remains compatible with a finite value of $\bar{R}^{1,1}$. In contrast, if $\bar{R}^{1,1}$ is infinite, then $\bar{R}^{1,2}$ is necessarily infinite, because the infinite number of events of the first type trigger an infinite number of events of the second type, since $n_{12} \neq 0$, even if self-triggering of events of type two is zero ($n_{22} = 0$). Moreover, a main peculiarity of degenerate (independent or semi-independent) systems is a strongly rectangular critical curve.

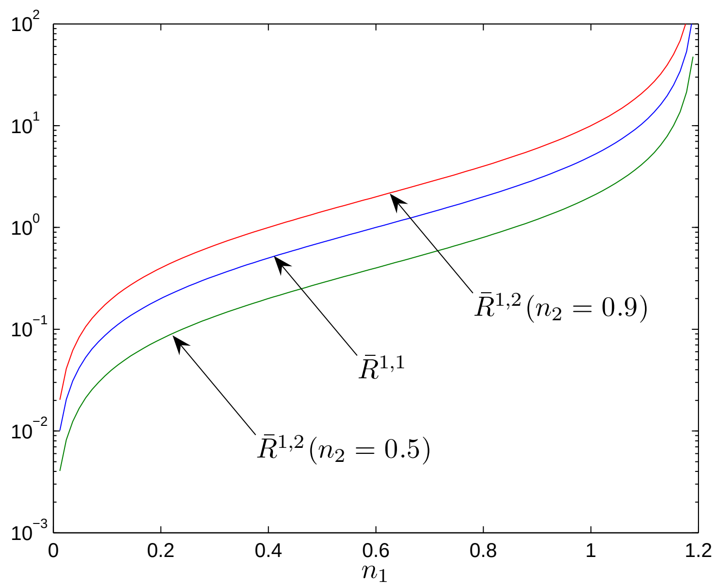

Fig. 4: (color online) Linear-log dependence of the mean numbers $\bar{R}^{1,1}$ and $\bar{R}^{1,2}$ given by (36) as a function of n_1, for $q = 0.2$ and for two values $n_2 = 0.5$ and $n_2 = 0.9$

Figure 4 shows in linear-log scale the number $\bar{R}^{1,1}$ (respectively $\bar{R}^{1,2}$) of events of the first type (respectively second type) over all generations that are triggered by a mother event of the first type, given by (36), as functions of n_1, for $q = 0.2$ and for two values $n_2 = 0.5$ and $n_2 = 0.9$. One can observe that, for $n_2 = 0.9$, $\bar{R}^{1,2}$ is larger than $\bar{R}^{1,1}$ for any $n_1 \in (0, 1.2)$.

D. One-dimensional chain of directed triggering

The case of a strong asymmetry in mutual triggering discussed in subsection IV C 2 for the two-dimensional case can be generalized to the case of $m > 2$ different event types. We consider a chain of directed influences $k \to k+1$ where the events of type k trigger events of both types k and $k+1$ only, and this for $k = 1, 2, ..., m$. This is captured by a form of the matrix $\hat{N}$ which has only the diagonal and the line above the diagonal with non-zero elements.

As the simplest example, consider first the matrix $\hat{N}$

$$\hat{N} = \begin{bmatrix} \chi & s & 0 & 0 & 0......... & 0... \\ 0 & \chi & s & 0 & 0......... & 0... \\ 0 & 0 & \chi & s & 0......... & 0... \\ 0 & 0 & 0 & \chi & s......... & 0... \\ \multicolumn{6}{c}{..................................} \end{bmatrix} \tag{37}$$

where

$$\chi = \frac{n}{1+q} \; , \qquad s = \frac{nq}{1+q} \; . \tag{38}$$

Reduced to a two-dimensional system $m = 2$, this corresponds to the particular case of subsection IV C 2 for which $n_{2,2} = n_2 = n_1/(1+q)$ in the notations of expressions (35), because the diagonal elements are taken all equal. For $m = 3$, a financial example is that the events of the first type correspond to fundamental news, the events of the second type are the price jumps of a leading market such as the US (assuming no feedbacks of prices on news) and the events of the third type correspond to the price jumps of a secondary market, such as the Russian stock market (assuming to effect of the Russian market on the US market).

Assuming that the mother event is of type k, the solution of equation (18) for this case (37) with (38) is given by $\bar{R}^{s,s} = \bar{R}^{k,s} = \bar{R}^{s,k} = 0$ for $1 \leq s < k$ and

$$\begin{aligned} \bar{R}^{k,k} &= \frac{\chi}{1-\chi} = \frac{n}{1+q-n}, \\ \bar{R}^{k,s} &= \frac{s^{s-k}}{(1-\chi)^{s-k+1}} = (1+q)\frac{(nq)^{s-k}}{(1+q-n)^{s-k+1}}, \qquad s > k \; . \end{aligned} \tag{39}$$

The fact that the critical point is solely controlled by a single critical value $n_c = 1/(1+q)$ results from the fact that all diagonal elements of the matrix (37) are equal.

E. One-dimensional chain of nearest-neighbor triggering

A natural extension to the above one-dimensional chain of directed triggering discussed in section IV D is to include the possibility of feedbacks from events of type $k+1$ to type k. The simple example is to consider symmetry mutual excitations confined to nearest neighbors in the sense of event types: $k \leftrightarrow k+1$. Mathematically, this is described by a symmetric matrix $\hat{N}$ of the average numbers $n_{k,s}$ of first-generation events of different types triggered by a mother of a fixed type. Figure 5 provides the geometrical sense of matrix $\hat{N}$ (40) for $m = 6$, where the circles represent the six types of events and the arrows denote their mutual excitation influences.

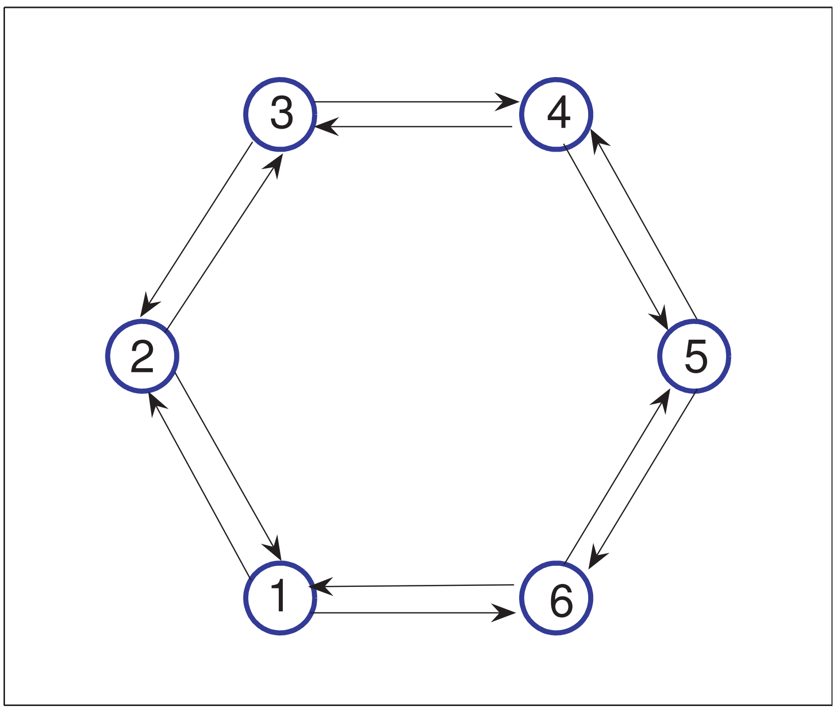

Fig. 5: Geometric sense of the matrix $\hat{N}$ for a one-dimensional chain of nearest-neighbor triggering.

Here, we consider that all diagonal elements are equal to some constant χ (same self-triggering abilities) and all off-diagnoal elements are equal to some different constant s (same mutual triggering abilities). The elements $n_{1,m}$ and $n_{m,1}$ are also equal to s to close the chain of mutual excitations between events of type 1 and of type m. Restricting to

$m = 6$ for illustration purpose, the corresponding matrix $\hat{N}$ reads

$$\hat{N} = \begin{bmatrix} \chi & s & 0 & 0 & 0 & s \\ s & \chi & s & 0 & 0 & 0 \\ 0 & s & \chi & s & 0 & 0 \\ 0 & 0 & s & \chi & s & 0 \\ 0 & 0 & 0 & s & \chi & s \\ s & 0 & 0 & 0 & s & \chi \end{bmatrix} \tag{40}$$

where

$$\chi = \frac{n}{1+q} \; , \qquad s = \frac{nq}{2(1+q)} \qquad \Rightarrow \qquad \chi + 2s = n \; . \tag{41}$$

As before, the parameter q quantifies the "strength" of the interactions between events of different types. Here, n represents the total number of first-generation events of all types that are generated by a given mother of fixed arbitrary type.

The solution of equation (18) for this case is given by a circulant structure

$$\hat{R} = \begin{bmatrix} A & B & C & D & C & B \\ B & A & B & C & D & C \\ C & B & A & B & C & D \\ D & C & B & A & B & C \\ C & D & C & B & A & B \\ B & C & D & C & B & A \end{bmatrix} \tag{42}$$

where

$$\begin{aligned} A(n,q) &= \frac{4n(1-n+q)^3 - (1-n+q)(5n-2q-2)n^2q^2 - n^4q^4}{4(1-n+q)^4 - 5(1-n+q)^2n^2q^2 + n^4q^4} \; , \\ B(n,q) &= \frac{nq(1+q)(2(1-n+q)^2 - n^2q^2)}{4(1-n+q)^4 - 5(1-n+q)^2n^2q^2 + n^4q^4} \; , \\ C(n,q) &= \frac{n^2q^2(1+q)(1-n+q)}{4(1-n+q)^4 - 5(1-n+q)^2n^2q^2 + n^4q^4} \; , \\ D(n,q) &= \frac{n^3q^3(1+q)}{4(1-n+q)^4 - 5(1-n+q)^2n^2q^2 + n^4q^4} \; . \end{aligned} \tag{43}$$

It is straightforward to check that the common denominator $4(1-n+q)^4 - 5(1-n+q)^2n^2q^2 + n^4q^4$ to these four numbers $A(n,q), B(n,q), C(n,q)$ and $D(n,q)$

1. does not vanish for any q values for $n < 1$,

2. vanishes for any q values at $n = 1$, and

3. can vanish at up to four values of q for $n > 1$.

This means that the system resulting from the structure of mutual excitations between different event types represented by the matrix (42) is always in the subcritical regime for $n < 1$ and becomes critical at $n = 1$ as for the decoupled or monovariate case.

The behavior of the ratios

$$\mathcal{R}_2 = \frac{\bar{R}^{1,2}}{\bar{R}^{1,1}} = \frac{B}{A}, \quad \mathcal{R}_3 = \frac{\bar{R}^{1,3}}{\bar{R}^{1,1}} = \frac{C}{A}, \quad \mathcal{R}_4 = \frac{\bar{R}^{1,4}}{\bar{R}^{1,1}} = \frac{D}{A} \tag{44}$$

is shown in figure 6 as a function of n for $q = 0.1$. Here, the mother event is of type 1 and the curves illustrate the progressing dampening of the cascade of triggering proceeding from type to type via nearest-neighbor mutual excitations.

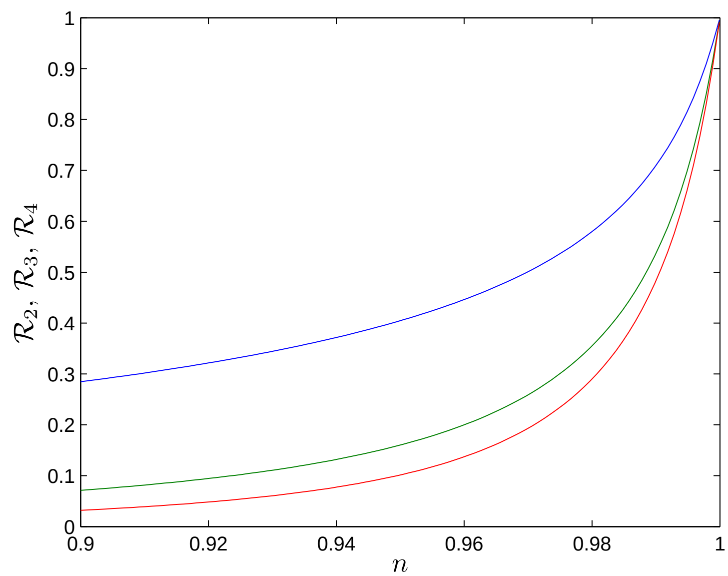

Fig. 6: (color online) Top to bottom: ratios $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ defined in (44) of the mean numbers of events of all generations generated by a mother event of type 1 for $q = 0.1$.

F. Subcriticality measure in the case of two types of events

For a system with two types of events, the explicit relation defining the critical curve is

$$n_2 = \mathcal{G}(n_1, q_1, q_2) = \frac{(1+q_1)(1+q_2) - (1+q_2)n_1}{(1+q_1) - (1-q_1 q_2)n_1}, \qquad 0 \leqslant n_1 \leqslant 1 + q_1, \tag{45}$$

which is represented in figure 7. One can verify that the point $(n_1 = 1, n_2 = 1)$ is always on the critical curves.

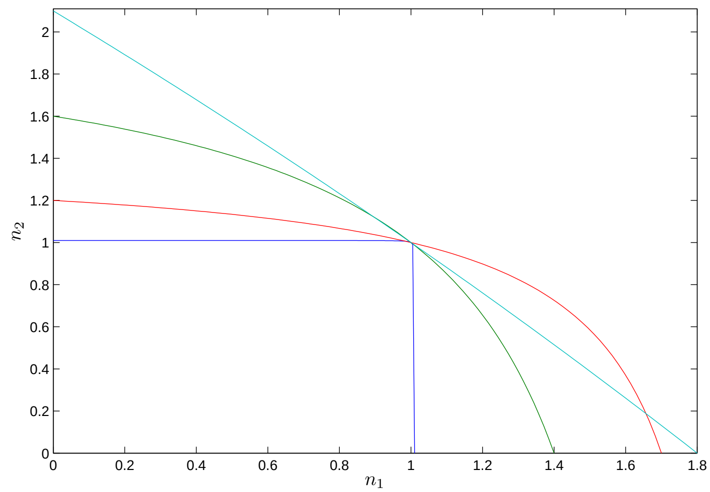

Fig. 7: (color online) Plots of critical curves $n_2(n_1)$ for systems with two types of events, for the following pairs of parameters: bottom up on the left side of the curves, we have $(q_1 = 0, q_2 = 0)$, $(q_1 = 0.4, q_2 = 0.6)$, $(q_1 = 0.7, q_2 = 0.2)$ and for $(q_1 = 0.8, q_2 = 1.1)$.

One can observe that, for $q_1 > 0$ and $q_2 > 0$, the subcritical domain is significantly larger than for systems in which event types are independent, i.e., do not mutually trigger each other, corresponding to $q_1 = q_2 \equiv 0$. It is illuminating to introduce a quantitative measure of the domain of subcriticality, here chosen for the two-dimensional case as the surface $\mathcal{S}$ of the subcritical domain:

$$\mathcal{S}(q_1, q_2) = \int_0^{1+q} \mathcal{G}(n_1, q_1, q_2) dn_1 \ . \tag{46}$$

For independent types $(q_1 = q_2 \equiv 0)$, $\mathcal{S}(0, 0) = 1$. The general expression of the subcriticality measure $\mathcal{S}(q_1, q_2)$ is obtained as

$$\mathcal{S}(q_1, q_2) = (1 + q_1)(1 + q_2) \frac{1 - q_1 q_2 \left[1 - \ln(q_1 q_2)\right]}{(1 - q_1 q_2)^2} \ . \tag{47}$$

In the particular case of a chain of directed triggering in the space of event types studied in subsection IV D for which the critical curve is rectangular, we find

$$\mathcal{S}(q) := \lim_{q_2 \to 0_+} \mathcal{S}(q, q_2) = 1 + q \ . \tag{48}$$

In the case of symmetric triggering among different event types ($q_1 = q_2 = q$), we obtain

$$\mathcal{S}(q,q) = \frac{1 - q^2 + q^2 \ln(q^2)}{(1-q)^2} \ . \tag{49}$$

These last two functions $\mathcal{S}(q)$ and $\mathcal{S}(q,q)$ are depicted in figure 8. The main insight is that mutual triggering tends to provide a significant extension of the stability domain.

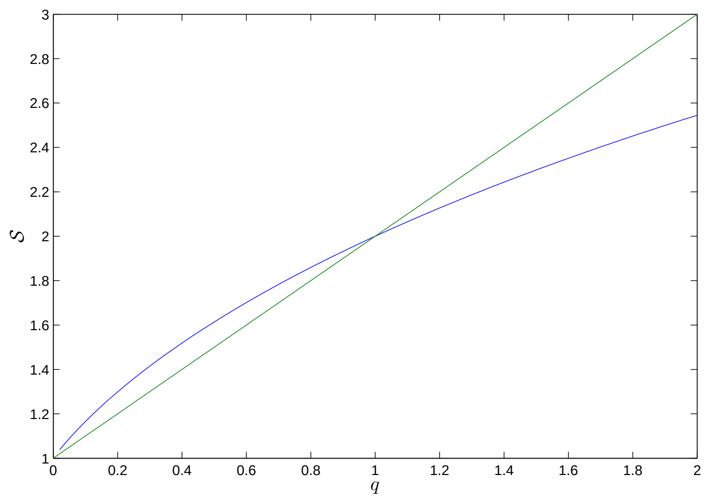

Fig. 8: (color online) Dependence of the subcriticality measures $\mathcal{S}(q)$ and $\mathcal{S}(q,q)$ given respectively by expression (48) (straight line) and (49) (concave curve).

V. CONCLUSION

Considering the class of multivariate self-excited Hawkes point processes, we have presented the general theory of multivariate generating functions to derive the number of events over all generations of various types that are triggered by a mother event of a given type. This has allowed us to discuss in details the stability domains of various systems, as a function of the topological structure of the mutual excitations across different event types. In particular, we have studied the case of symmetric mutual excitation abilities between events of different types, the case of just two different types of events, the case of a one-dimensional chain of directed triggering in the space of event types and the case of a one-dimensional chain in the space of event types with nearest-neighbor triggering. The main insight is that mutual triggering tends to provide a significant extension of the stability (or subcritical) domain compared with the case where event types are decoupled, that is, when an event of a given type can only trigger events of the type.

Acknowledgement: We acknowledge financial support from the ETH Competence Center "Coping with Crises in Complex Socio-Economic Systems" (CCSS) through ETH Research Grant CH1-01-08-2 This work was also partially supported by ETH Research Grant ETH-31 10-3.

[1] Brémaud, P., Point Processes and Queues, Springer, New York (1981).

[2] Daley, D.J. and D. Vere-Jones, An Introduction to the Theory of Point Processes, Springer Series in Statistics (2007).

[3] Horowitz, Paul and Winfield Hill, The Art of Electronics, 2nd edition. Cambridge (UK): Cambridge University Press, 1989, pp. 431-2.

[4] Montroll, E.W. and H. Scher, J. Stat. Phys., 9 (2), 101-135 (1973).

[5] Scher, H. and E.W. Montroll, Phys. Rev. B 12, 2455 (1975).

[6] Cont, Rama and Peter, Tankov, Financial Modelling with Jump Processes, Chapman & Hall/CRC Financial Mathematical Series, 2004

[7] Hawkes, A.G., *Journal of Royal Statistical Society, series B*, *33*, 438-443, 1971.

[8] Hawkes, A.G., Spectra of some mutually exciting point processes with associated variables, In Stochastic Point Processes, ed. P.A.W. Lewis, Wiley, 261-271, 1972.

[9] Hawkes, A.G. and Adamopoulos, L., Bull Internat. Stat. Inst. 45, 454-461, 1973.

[10] Hawkes, A.G. and Oakes D., Journal Apl. Prob. 11, 493-503, 1974.

[11] Ogata, Y., J. Am. stat. Assoc. 83, 9-27, 1988.

[12] Ogata, Y., Tectonophysics, 169, 159-174, 1989.

[13] Ogata, Y., Ann. Inst. stat. Mech. 50, 379-402, 1998.

[14] Ogata, Y., Pure Appl. Geophys. 155, 471-507, 1999.

[15] A. Helmstetter and D. Sornette J. Geophys. Res. 107, NO. B10, 2237, doi:10.1029/2001JB001580, 2002.

[16] A. Saichev and D. Sornette J. Geophys. Res., 112, B04313, doi:10.1029/2006JB004536, 2007.

[17] D. Sornette and I. Osorio, Prediction, chapter in "Epilepsy: The Intersection of Neurosciences, Biology, Mathematics, Physics and Engineering", Editors: Osorio I., Zaveri H.P., Frei M.G., Arthurs S., CRC Press, Taylor & Francis Group (2010) (http://arxiv.org/abs/1007.2420).

[18] V. Chavez-Demoulin, A.C. Davison and A.J. McNeil, Quantitative Finance, 5 (2), 227-234, 2005

[19] L. Bauwens and N. Hautsch, Modelling Financial High Frequency Data Using Point Processes, Handbook of Financial Time Series, Part 6, 953-979, DOI:10.1007/978-3-540-71297-8_41, 2009.

[20] E. Errais, K. Giesecke and L.R. Goldberg, Affine Point Processes and Portfolio Credit Risk (June 7, 2010). Available at SSRN: http://ssrn.com/abstract=908045.

[21] S. Azizpour, K. Giesecke and G. Schwenkler, Exploring the Sources of Default Clustering, working paper, Stanford University, 2010

[22] Y. Aït-Sahalia, J. Cacho-Diaz and R.J.A. Laeven, Modeling financial contagion using mutually exciting jump processes, Working Paper 15850, http://www.nber.org/papers/w15850, 2010.

[23] M. Salmon and W. W. Tham, Preferred Habitat, Time Deformation and the Yield Curve , working paper (Revised and resubmit - Journal of Financial Markets), 2008.

[24] A. Saichev and D. Sornette, Hierarchy of Temporal Responses of Multivariate Self-Excited Epidemic Processes, submitted to Physical Review E (http://arxiv.org/abs/1101.1611)

[25] Zhuang J., Vere-Jones D., Guan H., Ogata Y. and Ma L., Pure and Applied Geophysics, 162, 1367-1396, doi:10.1007/s00024-004-2674-3, 2005.

[26] T.J. Liniger, Multivariate Hawkes Processes, PhD Diss. ETH No. 18403, ETH Zurich, 2009.

[27] Helmstetter, A., Phys. Res. Lett. 91, 058501, 2003.

[28] D. Sornette and M.J. Werner, J. Geophys. Res. 110, No. B8, B08304, doi:10.1029/2004JB003535, 2005.

[29] D. Sornette and A. Helmstetter, Physical Review Letters 89 (15) 158501, 2002.

[30] A. Helmstetter and D. Sornette, Physical Review E. 66, 061104, 2002.

[31] A. Helmstetter, D. Sornette and J.-R. Grasso, J. Geophys. Res., 108 (B10), 2046, doi:10.1029/2002JB001991, 2003.

[32] A. Helmstetter and D. Sornette, Geophys. Res. Lett. 30 (11) doi:10.1029/2003GL017670, 2003.

[33] A. Helmstetter and D. Sornette, J. Geophys. Res., 108, 2482, 10.1029/2003JB002485, 2003.

[34] A. Saichev and D. Sornette, Phys. Rev. E 71, 056127, 2005.

[35] A. Saichev, A. Helmstetter and D. Sornette, Pure and Applied Geophysics 162, 1113-1134, 2005.

[36] A. Saichev and D. Sornette, Phys. Rev. E 70, 046123, 2004.

[37] A. Saichev and D. Sornette, Eur. Phys. J. B 51 (3), 443-459, 2006.

[38] D. Sornette, S. Utkin and A. Saichev, Physical Review E 77, 066109, 2008.

[39] Ozaki, T., Ann. Inst Statist. Math. 31, Part B, 145-155, 1979.

[40] Ogata, Y., IEEE Transactions on Information Theory, Vol.IT-27, No.1, Jan., 23-31, 1981.

[41] Ogata, Y. and Akaike, H., Journal of the Royal Statistical Society, Series B 44, No.1, 102-107,

1982.

[42] Ogata, Y., Journal of Physics of the Earth 31, 115-124, 1983.

[43] Zhuang J., Ogata Y. and Vere-Jones D., Journal of the American Statistical Association, 97, 369-380 (2002).

[44] Zhuang, J., Ogata, Y. and Vere-Jones, D., Journal of Geophysical Research 109 (B5), B05301, doi:10.1029/2003JB002879, 2004.

[45] Ogata Y. and Zhuang J., Tectonophysics, 413, 13-23, 2006.

[46] Zhuang J., PhD thesis, Department of Statistical Science, The Graduate University for Advanced Studies (2002).

[47] Marsan D. and O. Lengliné, Science 319 (5866), 1076-1079, 2008.

[48] D. Sornette and S. Utkin, Physical Review E 79, 061110, 2009.

[49] Harris, T.E.,The theory of branching processes, Springer, Berlin, 1963.

[50] D. Sornette, F. Deschatres, T. Gilbert and Y. Ageon, Phys. Rev. Letts. 93 (22), 228701, 2004.

[51] F. Deschatres and D. Sornette, Phys. Rev. E 72, 016112, 2005.

[52] D. Sornette, Endogenous versus exogenous origins of crises, in the monograph entitled "Extreme Events in Nature and Society," Series: The Frontiers Collection, S. Albeverio, V. Jentsch and H. Kantz, eds. (Springer, Heidelberg, 2005), pp 95-119, http://arxiv.org/abs/physics/0412026

[53] R. Crane and D. Sornette, Proc. Nat. Acad. Sci. USA 105 (41), 15649-15653, 2008.